Cluster structures on Grothendieck rings.

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We briefly review the classical concept of Grothendieck groups.Let $\mathcal{A}$ be an abelian category which is notherian and artinian. Then the Grothendieck group $K_0(\mathcal{A})$ has a canonical basis which consists of the classes of simple objects. On the other hand, if the abelian category $\mathcal{A}$ has a monoidal structure, the Grothendieck group $K_0(\mathcal{A})$ acquires a natural ring structure. A classical example of an abelian category, which has both properties is the category of finite dimensional representations of an Hopf algebra.Cluster algebras where introduced by Fomin and Zelevinsky around 2002 as a combinatorial framework to study dual canonical bases and total positivity in algebraic Lie theory. Hernandez and Leclerc introduced the notion of monoidal categorification of cluster algebras: Roughly speaking, in this case a cluster algebra A is realized as the Grothendieck ring $K_0(\mathcal{A}) = A$ of a monoidal, noetherian and and artinian abelian category $\mathcal{A}$ in such a way, that the cluster monomials are classes of (real) simple objects. This notion is motivated by the study of representations of quantum affine algebras and related structures. Conversely,this can be seen as an additional structure on the Grothendieck ring.We will review several examples of Grothendieck rings with a cluster algebra structure in this sense and point out how it leads to new formulas for q-characters of the simple representations which correspond to cluster monomials. Finally, I will report on an ongoing project with B. Leclerc and D. Hernandez, where we are developing a cluster algebra structure on the Grothendieck ring $K_0(\mathcal{O}^{\mathrm{sh}}_{\mathbb{Z}})$ of the (integral) category $\mathcal{O}^{\mathrm{sh}}_{\mathbb{Z}} = \oplus_{\mu \in P^\vee} \mathcal{O}^{\mu}_{\mathbb{Z}}$ for the shifted quantum affine algebras $U^{\mu}_{q}(L\mathfrak{g})$ associated to a complex simple Lie algebra $\mathfrak{g}$.

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