

From Markov to del Pezzo through Wahl

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In 1879, Andrey Markov published his master's thesis on binary quadratic forms. A central ingredient was the set of integral solutions to the equation $x^2 + y^2 + z^2 = 3xyz$, now known as the Markov equation. On the other hand, in 1887, Pascuale del Pezzo published an article that studied surfaces of degree d in projective space of dimension d . These surfaces are now referred to as del Pezzo surfaces, characterized by having an ample anticanonical class. The degree d of these surfaces ranges from 1 to 9, with a del Pezzo surface of degree 9 being equivalent to the projective plane.

During the 1980s, Wahl, Kollár–Shepherd-Barron, Kawamata studied certain quotient 2-dimensional singularities which appear on mild degenerations of surfaces. They are now referred to as Wahl singularities. Notably, parallel related developments occurred during the 1980s and 1990s in the context of smooth 4-manifolds, initiated by Fintushel–Stern and Park, as well as in the study of exceptional vector bundles by researchers such as Rudakov, Gorodentsev, Kuleshov, and Orlov. Following the work of Bădescu (1986), Manetti (1991), and Hacking (2001), Hacking and Prokhorov (2010) classified all degenerations of the complex projective plane with Wahl singularities. They proved that these degenerations can be fully understood through weighted projective planes $P(a^2, b^2, c^2)$ where (a, b, c) satisfies the Markov equation.

In a recent joint work with Juan Pablo Zúñiga <https://arxiv.org/abs/2504.19929>, we classified all Wahl singularities that appear in degenerations of del Pezzo surfaces for any fixed degree between 1 and 9. For that, we introduce various artefacts such as marked Wahl chains and slidings. This talk will present all of that.

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