Variation of Canonical Heights of Subvarieties

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Let V be a projective variety defined over a number field K, and let $\phi : V \to V$ be an endomorphism. Suppose E is an element of $\mathrm{Pic}(V)$ such that $\phi^* E$ is isomorphic to $E^{\otimes} d$. Call and Silverman, in their seminal paper, built a canonical height function $\hat{h}_\phi : V \to \mathbb{R}_{\geq} 0$ satisfying

$$\hat{h}_\phi(\phi(x)) = d\hat{h}_\phi(x) \quad \text{and} \quad \hat{h}_\phi(x) = h(x) + O(1),$$

where h is a Weil height on V with respect to E.

Call and Silverman showed that this height is well-defined and satisfies these properties. This height function is remarkable because it characterizes the points x in V whose forward orbit under ϕ is finite, generalizing to all projective varieties the classical work of Neron and Tate on elliptic curves.

If E is ample and $d \geq 2$, we say that (ϕ, V, E) is a polarized arithmetic dynamical system. A single dynamical system is well understood, but much less is known for families. This is the context of our work.

Let (f, X, L) be a family of polarized dynamical systems over a curve B, and let $P : B \to X$, where we denote $P(t)$ by P_t. On all but finitely many fibers (f_t, X_t, L_t), we may construct a canonical height $\hat{h}_{f_t}$.

Theorem 1 (Ingram, 2021) Let (f, X, L) be a family of polarized dynamical systems over a rational curve B, defined over $\mathbb{Q}$. Fix a degree one Weil height h_B on B. Then

$$\hat{h}_{f_t}(P_t) = \hat{h}_f(P) h_B(t) + O(h_B(t)^{1/2}).$$

Now assume L is equipped with an adelic continuous metric $\{\|\cdot\|_{v,i}\}$ for v in M_K, and denote the pair by $\bar{L} := (L, \{\|\cdot\|_v\} \quad \text{for} \quad v \quad \text{in} \quad M_K)$. When $\bar{L}$ is a big and nef $\mathbb{Q}$-line bundle endowed with a semi-positive continuous adelic

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metric, following Zhang (1995), we can define for any subvariety $Y \subseteq X$ of codimension q

$$h_{\bar{L}}(Y) := ((\bar{L}^(q+1)|Y))/((q+1)[K:\mathbb{Q}]\deg_Y(L)).$$

We prove the following theorem: Theorem There exists $\varepsilon > 0$ such that

$$\hat{h}_{f_t}(Y_t) = \hat{h}_f(V)h_B(t) + O(h_B(t)^{1-\varepsilon}).$$

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