Homotopy Covers of Graphs

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A covering map in topology is a surjective local homeomorphism, and such a map allows unique lifting of paths and homotopies between them. This notion has been translated to graphs using local isomorphisms, and a covering homomorphism of graphs is usually defined as a surjective homomorphism which is an isomorphism on neighbourhoods of vertices. With this definition, we get unique lifting of walks in graphs. However, once we introduce a notion of homotopy of graph homomorphisms, we diverge from the topological picture and coverings of graphs do not allow lifting of homotopies. In this work, we focus on X-homotopy between graph homomorphisms and consider what additional conditions are required to allow the lifting of homotopies between walks. Simply looking at neighbourhoods of vertices is insufficient here, and we define a homotopy covering map of graphs to be a surjective graph homomorphism which is an isomorphisms of 2-neighbourhoods around each vertex, extending our locality out to include neighbours of neighbours of vertices. These homotopy covering maps have been defined and studied in several other ways, by Tardiff-Wroncha and Matsushita. In this work we further develop the theory of homotopy covering maps, generalizing to graphs with multiple edges and loops. It turns out that multiple edges do not contribute interesting homotopy, but loops do; we investigate the patterns that result. We also generalize existing path lifting results about homotopy covers to a homotopy lifting property for arbitrary graph homomorphisms. This talk is based on joint work with T. Chih.

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