Krein-Sobolev Orthogonal Polynomials

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For fixed $\lambda > 0$, the polynomials $\{A_n\}_{n=0}^{\infty}$ orthogonal with respect to the inner product defined in the Sobolev space $H^1[-1,1]$

$$\langle f, g\rangle_\lambda := \int_{-1}^{1} (f(x)\overline{g}(x) + \lambda f'(x)\overline{g}'(x))dx$$

are called *Althammer's polynomials* or *Sobolev-Legendre polynomials.* They were first studied in 1962 by P. Althammer. Here, we present the orthogonal polynomials $\{K_n\}_{n=0}^{\infty}$ with respect to a (positive-definite) inner product as result of a left-definite spectral study of the self-adjoint Krein-Laplacian operator in $L^2(-1,1)$. We call these polynomials the *Krein-Sobolev polynomials.* The inner product $(\cdot,\cdot)_1$ naturally emerges from a left-definite spectral study of the one-dimensional shifted Krein Laplacian operator S_K. We review some properties of S_K and we establish various properties of the Krein-Sobolev polynomials, including completeness as well as the reality and simplicity of their roots in $(-1,1)$., and find an explicit formula for the polynomial family.

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