Cages and Choosing with Symmetries

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(Definitions at the end).

First the good news: In the work we present here, we improved five lower bounds for the order of cages: n(8,5) >= 69 (previously 67), n(5,7) >= 110 (prev. 108), n(4,9) >= 165 (prev. 162), n(3,14) >= 262 (prev. 260) and n(3,15) >= 388 (prev. 384).

We did this using novel computational search techniques for exhaustive searches in combinatorial spaces. Our main contribution is a new algorithm, Choose (S, k, G), which (given a set S, an integer k and a group G of permutations of S) computes all the k-subsets of S up to the symmetries specified by G. That is, Choose (S, k, G) produces a list of k-subsets of S which contains exactly one representative of each of the equivalence classes induced by the action of G on the set all k-subsets of S.

This algorithm can be used in a large class of exhaustive search problems. As a test problem, we tried it on the problem of searching for cages. We managed to reproduce most of the known results and also we improved the five lower bound for cages mentioned above. The known results that we could not reproduce are either a few infinite families known to exist for theoretical reasons, or cases where previous computational approaches took years of cpu time to complete (run in a few weeks on a mixture of machines). We do not have such computing power at our disposal at the present time.

– Definitions: An (d, g)-cage is a d-regular graph of girth g and minimum order. Cages always exist for $d \geq 2$ and $g \geq 3$. We denote by $n(d, g)$ the order of a (d, g)-cage.

Let S be a set and G a group of permutations of S. Given s in S and g in G, we denote by s^g the action of the permutation g on the element s. Let K the set of all subsets of S of cardinality k. Then G also acts on K,

MCA 2025 internal code: SS17-T14

since any m in K is of the form $m = \{s_1, s_2, \ldots, s_k\}$ and a permutation g in G acts on m element by element, that is, $m^g := \{s_1^g, s_2^g, ...s_k^g\}$. This induces an equivalence relation on K: two members m_1, m_2 of K, are equivalent if and only if there is a g in G such that $m_1 = m_2^g$.

In Special Session 17 on Graph Theory and its Applications at the Mathematical Congress of the Americas 2025.