Noetherian property in Melnikov functions

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This is a joint work with Pavao Mardesic, Dmitry Novikov and Laura Ortiz-Bobadilla.Let F be a polynomial, $F \in \mathbb{R}[x, y]$, and let $\mathcal{F}_\epsilon$ be a foliation in $\mathbb{C}^2$ defined by a deformation $dF + \epsilon\eta = 0$, where η is a polynomial one-form and ϵ is a small parameter. Assume that the foliation $dF = 0$ in $\mathbb{R}^2$ has a continuous family of periodic orbits $\gamma(t) \subset \{F = t\}$. The infinitesimal version of Poincaré center-focus problem and of the Hilbert's 16th problem, focus on determining the behavior of the family $\gamma(t)$ after the deformation $\mathcal{F}_\epsilon$; that is, whether the curves of the family $\gamma(t)$ after perturbation will remain as periodic orbits or if some limit cycles are born. To study this problem one considers the first return map with respect to γ and to the foliation $\mathcal{F}_\epsilon$:

$$P^{\gamma}_{\mathcal{F}_\epsilon}(t, \epsilon) = t + \epsilon M_1(t) + \epsilon^2 M_2(t) + \cdots .$$

The functions M_j are call Melnikov funtions, and they are univalued in a neighborhood of a regular value of F, but multivalued in $\mathbb{C}$ minus a finite set of values.The Melnikov functions contain essential information about what happens with the deformation of the continuous family $\gamma(t)$. From Françoise algorithm it is known that M_j are iterated integrals of length at most j. The complexity of these functions is codified by its length as iterated integrals. In this talk, we introduce the notion of length with respect to the orbit, and we will see that Melnikov functions can be expressed as evaluation of polynomials on a differential algebra. Then by the Ritt-Raundenbush theorem, some Noetherian properties will imply a drop on the length for all of the Melnikov functions. More precisely, we will see that for any $k \in \mathbb{N}$, there exists a $n = n(F, \gamma, k) \in \mathbb{N}$, depending only on F, γ and k, such that the identical vanishing of the first n Melnikov functions, will imply that the length with respect to the orbit of M_j is at most $j - k$, for every j.

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