

## The Semi-Discrete Lagrangian-Eulerian Method for Nonlocal Conservation Laws in Several Dimensions.

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In this talk, we will present the semi-discrete Lagrangian-Eulerian method to approximate a class of multi-dimensional scalar conservation laws with nonlocal flux, referred to as the nonlocal model:

$$\partial_t \rho(t, \mathbf{x}) + \sum_{i=1}^d \partial_{x_i} \left( V^i \left[ W[\rho, \omega](t, \mathbf{x}) \right] F^i(\rho(t, \mathbf{x})) \right) = 0, \quad (t, \mathbf{x}) \in (0, T) \times \mathbb{R}^d.$$

This model arises in several important applications, such as traffic flow, porous media, and biological systems, where nonlocal interactions significantly influence the dynamics. The nonlocal term  $W[\rho, \omega](t, \mathbf{x})$  introduces spatial averaging, which adds complexity to both the analysis and numerical approximation. We will analyze the convergence of this method using the weak asymptotic approach, previously developed in [1], and extend its results to the multidimensional nonlocal case. This technique is particularly well-suited for the Lagrangian-Eulerian schemes under consideration. The weak asymptotic method constructs a family of approximate solutions with the following key properties:

1. The approximate functions are uniformly bounded in  $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ ;
2. The family is governed by a suitable modulus of continuity in both time and space.

These properties allow us to apply an  $L^1$ -compactness argument, from which we can extract a convergent subsequence whose limit is a weak entropy solution of the original nonlocal conservation law. To complement the theoretical

findings, we will present several numerical examples. This talk presents results from [2] and was conducted in collaboration with E. Abreu, R. de la Cruz, J. Juajibioy, and W. Lambert.

## References

- [1] E. Abreu, M. Colombeau, E. Panov, Weak asymptotic methods for scalar equations and systems, *J. Math. Anal. Appl.* 444 (2016) 1203-1232.
- [2] E. Abreu, R. de la Cruz, J. Juajibioy, W. Lambert, Semi-discrete Lagrangian-Eulerian approach based on the weak asymptotic method for nonlocal conservation laws in several dimensions, *Journal of Computational and Applied Mathematics*. Volume 458, April 2025, 116325.

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