Approximate solutions to the superconducting interface model

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We consider the superconducting interface model introduced in Kyle Thompson's doctoral thesis from 2016. The most general version of this model is described by the semilinear system of PDEs

$$\begin{aligned} \varepsilon^2\left(\partial_{tt}\varphi - \Delta_x\varphi\right) + \lambda_\varphi\left(\varphi^2 - 1\right)\varphi + \beta|\sigma|^2\varphi = 0 \\ \varepsilon^2\left(\partial_{tt}\sigma - \Delta_x\sigma\right) + \lambda_\sigma\left(|\sigma|^2 - m_\sigma^2\right)\sigma + \beta\varphi^2\sigma = 0, \end{aligned} \tag{1}$$

where $0 < \varepsilon \ll 1$ and $(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0, \infty)^4$ are parameters and

$$\varphi : (0,T) \times \mathbb{R}^n \to \mathbb{R} \qquad \text{and} \qquad \sigma : (0,T) \times \mathbb{R}^n \to \mathbb{C}$$

are the unknowns, defined over $(0,T) \times \mathbb{R}^n$ for some $T > 0$ and $n \geq 2$. We are interested in the scenario where the domain can be partitioned as

$$(0,T) \times \mathbb{R}^n = \mathcal{O}^+ \cup \Gamma \cup \mathcal{O}^-,$$

where Γ is a timelike hypersurface representing the interface between the two disjoint connected open subsets $\mathcal{O}^+$ and $\mathcal{O}^-$, with respect to which φ and σ exhibit the following behaviour:

$$\varphi \approx \begin{cases} +1, & \text{in } \mathcal{O}^+ \\ -1, & \text{in } \mathcal{O}^- \end{cases} \qquad \text{and} \qquad |\sigma| \text{ is strictly positive and concentrated in an } \varepsilon\text{-neighbourhood of } \Gamma.$$

In the above context, σ represents a carrying-current field, and the target setting described above can be understood as a reduced version of the superconducting string model proposed by Edward Witten in 1984. The reduction

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is a result of the assumptions that φ is real valued and that there are no perceivable effects due to the presence of external electromagnetic fields.In this talk, we present a construction of approximate solutions to (1) exhibiting the desired characteristics outlined above. We will place particular emphasis on the conditions on the parameters $(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta)$ and on the pair (Γ, θ), where θ is the phase of σ, which make this construction possible. In particular, we discuss the one-dimensional stationary problem associated with (1), and prove the existence of an open set of parameters for which this problem has solutions qualitatively similar to $(\varphi_0, \sigma_0) = (\tanh, \operatorname{sech})$ that are instrumental in the construction of more intricate approximate solutions to (1). On the other hand, we derive the geometric quasilinear PDE which embodies the relationship between Γ and the phase of σ required for the construction of approximate solutions, and present a well-posedness result for these equations under a graph gauge condition on Γ.This is joint work with Robert Jerrard, Mauro Bonafini, and Giandomenico Orlandi.

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